The Hardy-Littlewood Circle Method: A Case Study in the Circulation of Ideas

Adrian Rice
Randolph-Macon College
The Circle Method

Suppose we wish to determine an arithmetic function \( r(n) \) that has generating function

\[
f(x) = \sum_{n=0}^{\infty} r(n) x^n,
\]

with radius of convergence equal to 1.
The Circle Method

Suppose we wish to determine an arithmetic function $r(n)$ that has generating function

$$f(x) = \sum_{n=0}^{\infty} r(n)x^n,$$

with radius of convergence equal to 1.

Then by the Cauchy Integral Formula,

$$r(n) = \frac{1}{2\pi i} \int_c \frac{f(x)}{x^{n+1}} \, dx$$

where $C$ is a closed path entirely within the unit circle $|x| = 1$. 
The value of the integral

\[ \frac{1}{2\pi i} \int_C \frac{f(x)}{x^{n+1}} \, dx \]

is largely determined by its residues at the roots of unity \( \omega_1, \omega_2, \omega_3, \ldots \)

\[ \omega_q = \exp \left( \frac{2\pi i}{q} \right), \]

where each successive residue \( R_1, R_2, R_3, \ldots \) contributes less and less.
The Circle Method

Thus, if the value of the integral

\[ \frac{1}{2\pi i} \int_C \frac{f(x)}{x^{n+1}} \, dx \sim F(n) \]

where

\[ F(n) = R_1 + R_2 + R_3 + \ldots \]

then

\[ r(n) \sim F(n). \]
G. H. Hardy  
(1877–1947)  

J. E. Littlewood  
(1885–1977)
G. H. Hardy
(1877–1947)
The Circle Method

- Hardy & Ramanujan (1916–18) on partitions.
The Circle Method

- Hardy & Ramanujan (1916–18) on partitions.
- Hardy (1918–20) on sums of squares.
The Circle Method

- Hardy & Ramanujan (1916–18) on partitions.
- Hardy (1918–20) on sums of squares.
- Hardy & Littlewood (1920–28) on Waring’s Problem.
The Circle Method

- Hardy & Ramanujan (1916–18) on partitions.
- Hardy (1918–20) on sums of squares.
- Hardy & Littlewood (1920–28) on Waring’s Problem.
- Hardy & Littlewood (1922–24) on the Goldbach Conjecture.
Partitions

Let $p(n)$ represent the number of ways a positive integer $n$ can be written as a sum of positive integers.

E.g. Let $n = 4$

\[= 3 + 1\]
\[= 2 + 2\]
\[= 2 + 1 + 1\]
\[= 1 + 1 + 1 + 1.\]

Therefore, $p(4) = 5.$
Partitions

Let $p(n)$ represent the number of ways a positive integer $n$ can be written as a sum of positive integers.

E.g. Let $n = 4$

\[
= 3 + 1 \\
= 2 + 2 \\
= 2 + 1 + 1 \\
= 1 + 1 + 1 + 1.
\]

Therefore, $p(4) = 5$.

But what about $p(100)$? Or $p(200)$?
**Partitions**

Euler (1748) had proved that the generating function of \( p(n) \) is

\[
\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}
\]
Hardy-Ramanujan (1916–18)

“Asymptotic formulae in Combinatory Analysis”,
Hardy-Ramanujan (1916–18)

\[ f(x) = \sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \ldots} \]
Hardy-Ramanujan (1916–18)

\[ f(x) = \sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)\ldots} \]

\[ = \frac{x^{1/24}}{\eta(x)} \]

where

\[ \eta(x) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n), \quad x = e^{\pi i \tau}, \quad \text{and} \quad Im(\tau) > 0. \]
Hardy-Ramanujan (1916–18)

\[ p(n) = \frac{1}{2\pi i} \int_C \frac{f(x)}{x^{n+1}} \, dx \]

where \( C \) is a closed path entirely within the unit circle enclosing the origin.
Hardy-Ramanujan (1916–18)

Let $A_k(n) = \sum_{h=1}^{k} \sum_{(h,k)=1} \omega_{h,k} e^{-2\pi i h n/k}$ with $\omega_{h,k} = \exp \{ \pi i s(h, k) \}$,

and let $c = \pi \sqrt{2/3}$, and $\lambda_n = \sqrt{n - 1/24}$.

Then:

$$p(n) \sim \frac{1}{2\pi \sqrt{2}} \left\lfloor \alpha \sqrt{n} \right\rfloor \sum_{k=1}^{\left[ \alpha \sqrt{n} \right]} A_k(n) \sqrt{k} \frac{d}{dn} \left( e^{c \lambda_n/k} \lambda_n \right)$$
Hardy-Ramanujan (1916–18)

- Hardy-Ramanujan formula: \( p(100) = 190,569,291.996 \)
Hardy-Ramanujan (1916–18)

- Hardy-Ramanujan formula: \( p(100) = 190,569,291.996 \)
- Precise value when \( n = 100 \): \( p(100) = 190,569,292 \)
Hardy-Ramanujan (1916–18)

- Hardy-Ramanujan formula: \( p(100) = 190,569,291.996 \)
- Precise value when \( n = 100 \): \( p(100) = 190,569,292 \)
- Hardy-Ramanujan formula: \( p(200) = 3,972,999,029,388.004 \)
Hardy-Ramanujan (1916–18)

- Hardy-Ramanujan formula: \( p(100) = 190,569,291.996 \)
- Precise value when \( n = 100 \): \( p(100) = 190,569,292 \)

- Hardy-Ramanujan formula: \( p(200) = 3,972,999,029,388.004 \)
- Precise value when \( n = 200 \): \( p(200) = 3,972,999,029,388 \)
Hardy-Ramanujan (1916–18)

Publication Strategy
Hardy-Ramanujan (1916–18)

Publication Strategy

Hardy-Ramanujan (1916–18)

**Publication Strategy**


- “Une formule asymptotique pour le nombre des partitions de $n$”, *Comptes Rendus* **164** (1917), 35-38.
Hardy-Ramanujan (1916–18)

Publication Strategy

• “Asymptotic formulae in combinatory analysis”, *Quatrième Congrès des Mathématiciens Scandinaves*, 1916, 45-53.

• “Une formule asymptotique pour le nombre des partitions de $n$”, *Comptes Rendus* **164** (1917), 35-38.

Hardy-Ramanujan (1916–18)

Publication Strategy


- “Une formule asymptotique pour le nombre des partitions de $n$”, *Comptes Rendus* 164 (1917), 35-38.

Rademacher (1937)

“On the partition function $p(n)$”,
\[ p(n) \sim \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\lfloor \alpha \sqrt{n} \rfloor} A_k(n) \sqrt{k} \frac{d}{dn} \left( \frac{e^{c\lambda_n/k}}{\lambda_n} \right) \]
Rademacher (1937)

$$p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left( \frac{\sinh(c\lambda_n/k)}{\lambda_n} \right)$$
Hardy-Ramanujan (1918)

“Our method may be applied in its full power to the study of the representation of a number as the sum of $s$ squares, and yields results very similar to those which we have found concerning $p(n)$.”
Partitions into Sums of Squares

Let $r_s(n)$ be the total number of representations of a positive integer $n$ as a sum of $s$ squares.

For example, since

$5 = (-2)^2 + (-1)^2$
$= 2^2 + (-1)^2$
$= (-2)^2 + 1^2$
$= 2^2 + 1^2$
$= (-1)^2 + (-2)^2$
$= 1^2 + (-2)^2$
$= (-1)^2 + 2^2$
$= 1^2 + 2^2$
Partitions into Sums of Squares

Let $r_s(n)$ be the total number of representations of a positive integer $n$ as a sum of $s$ squares.

For example, since

$$5 = (-2)^2 + (-1)^2$$
$$= 2^2 + (-1)^2$$
$$= (-2)^2 + 1^2$$
$$= 2^2 + 1^2$$
$$= (-1)^2 + (-2)^2$$
$$= 1^2 + (-2)^2$$
$$= (-1)^2 + 2^2$$
$$= 1^2 + 2^2$$

$r_2(5) = 8$
Partitions into Sums of Squares

Let $r_s(n)$ be the total number of representations of a positive integer $n$ as a sum of $s$ squares.

For example, since

\[
4 = (-2)^2 + 0^2 + 0^2 \\
= 0^2 + (-2)^2 + 0^2 \\
= 0^2 + 0^2 + (-2)^2 \\
= 2^2 + 0^2 + 0^2 \\
= 0^2 + 2^2 + 0^2 \\
= 0^2 + 0^2 + 2^2
\]
Partitions into Sums of Squares

Let \( r_s(n) \) be the total number of representations of a positive integer \( n \) as a sum of \( s \) squares.

For example, since

\[
4 = (-2)^2 + 0^2 + 0^2 \\
= 0^2 + (-2)^2 + 0^2 \\
= 0^2 + 0^2 + (-2)^2 \\
= 2^2 + 0^2 + 0^2 \\
= 0^2 + 2^2 + 0^2 \\
= 0^2 + 0^2 + 2^2 \\
r_3(4) = 6
\]
Lagrange (1770) proved that:

Every positive integer can be expressed as the sum of at most 4 squares.
Legendre (1798) proved that:

\[ r_3(n) = 0 \text{ if } n \equiv 7 \mod 8 \]
Partitions into Sums of Squares

Gauss (1801) proved that:

\[
r_3(n) = \begin{cases} 
24h(-n), & \text{if } n \equiv 3 \mod 8 \\
12h(-4n), & \text{if } n \equiv 1, 2, 5, 6 \mod 8 
\end{cases}
\]

where \( h(n) \) is the class number of \( n \).
Partitions into Sums of Squares

Jacobi (1829) proved that:

- \( r_2(n) = 4 \sum_{d | n} (-1)^{(d-1)/2} \) for \( d \) odd
- \( r_4(n) = 8 \sum_{d | n} d \) for \( 4 \nmid d \)
- \( r_6(n) = 16 \sum_{d | n} \chi(d')d^2 - 4 \sum_{d | n} \chi(d)d^2 \)
- \( r_8(n) = 16 \sum_{d | n} (-1)^{n+d}d^3 \)
Partitions into Sums of Squares

Smith (1867) and Minkowski (1882) proved that:

\[ r_5(n) = \frac{C n^{3/2}}{\pi^2} \sum_{\substack{m \text{ odd} \\ (m,n) = 1}} \left( \frac{n}{m} \right) \frac{1}{m^2} \]

where \( C = 80 \) \( (n \equiv 0, 1, 4 \mod 8) \),
\( C = 160 \) \( (n \equiv 2, 3, 6, 7 \mod 8) \),
\( C = 112 \) \( (n \equiv 5 \mod 8) \),

and \( \left( \frac{n}{m} \right) \) is a Legendre-Jacobi symbol.
Smith (1867) proved (for odd $n$) that:

$$r_7(n) = \frac{Cn^{5/2}}{\pi^3} \sum_{m \text{ odd} \atop (m,n)=1} \left( \frac{-n}{m} \right) \frac{1}{m^2}$$

where $C = 448 \ (n \equiv 1 \mod 8)$,

$C = 560 \ (n \equiv 3 \mod 8)$,

$C = 448 \ (n \equiv 5 \mod 8)$,

$C = 592 \ (n \equiv 7 \mod 8)$,

and $\left( \frac{-n}{m} \right)$ is a Legendre-Jacobi symbol.
Hardy (1918–20)

“On the Representation of a Number as the Sum of Any Number of Squares”,
Hardy (1918–20)

Publication Strategy

- “On the representation of a number as the sum of any number of squares, and in particular of five or seven”, *Proc. Nat. Acad. Sci.* 4 (1918), 189-93.

- “On the representation of a number as the sum of any number of squares, and in particular of five”, *Trans. Amer. Math. Soc.* 21 (1920), 255-84.
Hardy (1918–20)

Publication Strategy

- “On the representation of a number as the sum of any number of squares, and in particular of five or seven”, Proc. Nat. Acad. Sci. 4 (1918), 189-93.

- “On the representation of a number as the sum of any number of squares, and in particular of five”, Trans. Amer. Math. Soc. 21 (1920), 255-84.
Partitions into Sums of Squares

Jacobi (1829) had proved that the generating function of $r_s(n)$ is

$$f(x) = \sum_{n=0}^{\infty} r_s(n)x^n = [\theta(\tau)]^s = \left[ \sum_{n=-\infty}^{\infty} e^{\pi in^2 \tau} \right]^s$$
Hardy (1918–20)

\[ f(x) = \sum_{n=0}^{\infty} r_s(n)x^n = [\theta(\tau)]^s \]

where

\[ \theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau}, \quad x = e^{\pi i\tau}, \quad \text{and} \quad Im(\tau) > 0. \]
And using the identity \( \theta(\tau) = \sqrt{\frac{\tau}{i}} \theta \left( \frac{-1}{\tau} \right) \),

the Circle Method gave, for \( s = 5 \) and \( s = 7 \),
And using the identity $\theta(\tau) = \sqrt{\frac{i}{\tau}} \theta \left( \frac{-1}{\tau} \right)$, the Circle Method gave, for $s = 5$ and $s = 7$,

$$r_5(n) = \frac{An\sqrt{n}}{\pi^2} \sum_{\substack{m \text{ odd} \\ (m,n)=1}} \left( \frac{n}{m} \right) \frac{1}{m^2}$$

$$r_7(n) = \frac{Bn^2\sqrt{n}}{\pi^3} \sum_{\substack{m \text{ odd} \\ (m,n)=1}} \left( \frac{-n}{m} \right) \frac{1}{m^2}$$
And using the identity $\theta(\tau) = \sqrt{\frac{i}{\tau}} \theta \left( \frac{-1}{\tau} \right)$, the Circle Method gave, for $s = 5$ and $s = 7$,

$$r_5(n) = \frac{An \sqrt{n}}{\pi^2} \sum_{m \text{ odd} \atop (m,n)=1} \left( \frac{n}{m} \right) \frac{1}{m^2}$$

$$r_7(n) = \frac{Bn^2 \sqrt{n}}{\pi^3} \sum_{m \text{ odd} \atop (m,n)=1} \left( \frac{-n}{m} \right) \frac{1}{m^2}$$
And if $s \geq 9$, and \( A_k(n) = \sum_{\substack{h=1 \atop (h,k)=1}}^{k} e^{\pi i h n^2 / k} \),

then

\[
 r_s(n) \sim \frac{\pi^{s/2}}{\Gamma(s/2)} n^{s/2-1} \sum_{h,k} \left( \frac{A_k(n)}{k} \right)^s e^{-\pi i h n / k}
\]
“But with the entry of asymptotic formulas, the peculiar interest of squares as such departs, and the problem becomes merely a somewhat trivial case of the much larger problem usually described as Waring’s problem, and so of the investigations which Mr. Littlewood and I are publishing elsewhere.”
Waring’s Problem

In *Meditationes Algebraicae* (1770), Edward Waring conjectured that:

“Every integer is the sum of at most four squares, nine [positive] cubes, nineteen fourth powers, etc.”
Waring’s Problem

If we define a number $g(k)$ to be such that a positive integer can be written as the sum of at most $g(k)$ $k$th powers, then

$$g(2) = 4, \quad g(3) = 9, \quad g(4) = 19.$$
Waring’s Problem

Part One:

Does $g(k)$ exist for every positive integer?

Part Two:

What is the actual value of $g(k)$ as a function of $k$?
Waring’s Problem

**Part One:**

Does $g(k)$ exist for *every* positive integer?

**Part Two:**

What is the actual value of $g(k)$ as a function of $k$?
“Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl $n$-ter Potenzen (Waringsches Problem)”, *Mathematische Annalen* 67 (3) (1909), 281–300.
“Hilbert’s work … is absolutely and triumphantly successful, and it stands with the work of Hadamard and de la Vallee-Poussin … as one of the landmarks in the modern history of the theory of numbers. But there is an enormous amount which remains to be done, and it would seem that, if we are to interpret Waring’s problem in the widest possible sense, … then essentially different and inherently more powerful methods are required.”
Waring’s Problem

**Part One:**

Does $g(k)$ exist for *every* positive integer?

**Part Two:**

What is the actual value of $g(k)$ as a function of $k$?
Waring’s Problem

PART ONE:

Does \( g(k) \) exist for every positive integer?

PART TWO:

What is the actual value of \( g(k) \) as a function of \( k \)?
Waring’s Problem

Lagrange (1770): \( g(2) = 4 \)
Waring’s Problem

Lagrange (1770): $g(2) = 4$

Liouville (1859): $g(4) \leq 53$
Waring’s Problem

Lagrange (1770): $g(2) = 4$

Liouville (1859): $g(4) \leq 53$

Wieferich (1909): $g(3) = 9$

Kempner (1912): $g(3) = 9$
Weyl (1916)

“Über die Gleichverteilung von Zahlen mod. Eins”,
*Mathematische Annalen* 77 (1916), 313–352.
Hardy-Littlewood (1920)

Hardy and Littlewood introduced a number $G(k)$ to be such that every sufficiently large positive integer can be written as the sum of at most $G(k)$ $k$th powers.
Hardy-Littlewood (1920)

In other words, if $r_{k,s}(n)$ is the total number of ways that a positive integer $n$ can be written as the sum of $s$ positive $k$th powers, then

$$G(k) = \min \left\{ s : \lim_{n \to \infty} r_{k,s}(n) > 0 \right\}$$
In other words, if \( r_{k,s}(n) \) is the total number of ways that a positive integer \( n \) can be written as the sum of \( s \) positive \( k \)th powers, then

\[
G(1) = 1, \quad G(2) = 4, \quad \text{and in general}, \quad G(k) \leq g(k).
\]
For $k \geq 3$, let

$$f(x) = 1 + 2 \sum_{n=1}^{\infty} x^{nk}$$

so that

$$(f(x))^s = \sum_{n=0}^{\infty} r_{k,s}(n)x^n$$
Hardy-Littlewood (1920)

\[ r_{k,s}(n) = \frac{1}{2\pi i} \int_C \frac{(f(x))^s}{x^{n+1}} \, dx \]

where \( C \) is a closed path entirely within the unit circle enclosing the origin.
If \( S = \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left( \frac{S_{p,q}}{q} \right) e^{2\pi i np/q} \) with \( S_{p,q} = \sum_{k=0}^{q-1} e^{2\pi h^p k p \pi / q} \)

then

\[
r_{k,s}(n) \sim \frac{\left\{ 2\Gamma \left( 1 + \frac{1}{k} \right) \right\}^s}{\Gamma \left( 1 + \frac{s}{k} \right)} n^{(s/k)-1} S
\]
Hardy-Littlewood (1920)

Corollary:

“If $n$ is sufficiently large, then $r_{k,s}(n)$ cannot be zero, and representations of $n$ by $s$ $k$th powers certainly exist. The way is thus open to a proof of the existence of $G(k)$; if $G(k)$ exists, so also does $g(k)$, and Waring’s problem is solved.”
Hardy-Littlewood (1925)

Main Result:

$$G(k) \leq (k - 2)2^{k-1} + 5$$
Hardy-Littlewood (1920–28)

Publication Strategy


• “Some problems of ‘Partitio Numerorum’ II: Proof that every large number is the sum of at most 21 biquadrates”, Mathematische Zeitschrift 9 (1921), 14-27.


Hardy-Littlewood (1920–28)

Publication Strategy

- “Some problems of ‘Partitio Numerorum’ II: Proof that every large number is the sum of at most 21 biquadrates”, Mathematische Zeitschrift 9 (1921), 14-27.
“Of prime importance is the theory originated by Hardy and Littlewood, which applies not merely to Waring’s problem but also to various other problems in additive number theory, such as the theory of partitions and Goldbach’s and related theorems on sums of primes.”
Dickson (1933)

“The elaborate theory due to Hardy and Littlewood yields a number $C(s, n)$ beyond which every integer is a sum of $s$ integral $n$th powers greater than or equal to zero. Since $C$ is excessively large, their theory yields essentially only asymptotic theorems. For several years the writer has been elaborating his idea that it is possible to supplement these asymptotic theorems and show that they hold also for all integers below $C$. ”
James (1932)

James (1934)

“The value of the number $g(k)$ in Waring’s problem”, 
James (1934)

\[ g(6) \leq 183, \quad g(7) \leq 322, \quad g(8) \leq 595 \]
Dickson (1936)

“Solution of Waring’s problem”,
Dickson (1936)

\[ g(k) = 2^k + \left\lceil \left(\frac{3}{2}\right)^k \right\rceil - 2 \]

for \( k \geq 7 \).
Vinogradov (1928)

“О Теореме Варинга”,
Известия Академии Наук СССР (1928), 393–400.
“Despite the fact that much work has been devoted to Waring’s theorem, we consider the method proposed in the present paper to be not devoid of interest.

The most perfect results in this direction have so far been obtained in the excellent works of Hardy-Littlewood. However, even after simplifications, their method … involves considerable complexity.

In this paper, using in essence the same means as Hardy-Littlewood, … we get the same results with incomparably greater brevity and simplicity.”
Vinogradov (1935)

“On Waring’s problem”,
Vinogradov (1935)

\[ G(k) \leq 6k \log k + (4 + \log 216)k \]
Heilbronn (1935)

“Über das Waringsche Problem”,
*Acta Arithmetica* 1 (1935), 212–221.
Hua Luogeng (1938)

“On Waring’s Problem”,
Hua Luogeng (1938)

\[ G(5) \leq 28 \]
Davenport (1939)

“On Waring’s Problem for fourth powers”,
Davenport (1939)

\[ G(4) = 16 \]
The Goldbach Conjecture

“Our method is applicable in principle to this problem also. We cannot solve the problem, but we can open the first serious attack upon it, and bring it into relation with the established prime number theory.”
The Goldbach Conjecture

In a 1742 letter to Euler, Christian Goldbach made the conjecture that:

“Every integer greater than 2 can be written as the sum of three primes.”
The Goldbach Conjecture

**Strong Version:**
Every even number greater than 2 is expressible as the sum of two primes.

**Weak Version:**
Every odd number greater than or equal to 7 is expressible as the sum of three primes.
Hardy-Littlewood (1922–24)

Publication Strategy

• “Goldbach’s Theorem”, Matematisk Tidsskrift B (1922), 1-16.

• “Some problems of ‘Partitio Numerorum’ III: On the expression of a number as a sum of primes”, Acta Mathematica 44 (1922), 1-70.


Hardy-Littlewood (1922–24) Publication Strategy

• “Goldbach’s Theorem”, Matematisk Tidsskrift B (1922), 1-16.

• “Some problems of ‘Partitio Numerorum’ III: On the expression of a number as a sum of primes”, Acta Mathematica 44 (1922), 1-70.


Hardy-Littlewood (1922)

“Hypothesis R”:

There exists a real number $\Theta < \frac{3}{4}$ such that all zeros of all $L$-series $L(s, \chi)$ formed with Dirichlet characters lie in the half-plane $\sigma \leq \Theta$. 

Hardy-Littlewood (1922)
Hardy-Littlewood (1922)

Let \( N_3(n) \) be the number of representations of an odd positive integer \( n \) as a sum of three odd primes.

If 
\[
C_3 = \prod_q \left(1 + \frac{1}{(q - 1)^3}\right),
\]
where \( q \) runs through all odd primes,

then:
\[
N_3(n) \sim C_3 n^2 (\log n)^{-3} \prod_p \frac{(p - 1)(p - 2)}{p^2 - 3p + 3}
\]
where \( p \) runs through all odd prime divisors of \( n \).
Hardy-Littlewood (1922)

Corollary:

For sufficiently large $n$, $N_3(n) > 0$. 
Corollary:

For sufficiently large $n$, $N_3(n) > 0$.

i.e.

Every large odd number is the sum of 3 odd primes.
**Hardy-Littlewood (1922)**

**Corollary:**

For sufficiently large \( n \), \( N_3(n) > 0 \).

i.e.

Every large odd number is the sum of 3 odd primes.

“... an imperfect and provisional result, but it is the first serious contribution to the solution of the problem.”
Vinogradov (1937)

Estermann (1938)

Open Questions
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals?
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S. / Trans. A.M.S. / Math Zeitschrift / ...*
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S. / Trans. A.M.S. / Math Zeitschrift / ...*
  • Books?
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals?  *Proc. L.M.S.* / *Trans. A.M.S.* / *Math Zeitschrift* / ...
  • Books?  Landau’s *Vorlesungen über Zahlentheorie* (1927)
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S. / Trans. A.M.S. / Math Zeitschrift / ...*
  • Books? Landau’s *Vorlesungen über Zahlentheorie* (1927)
  • Graduate students?
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S.* / *Trans. A.M.S.* / *Math Zeitschrift* / ...
  • Books? Landau’s *Vorlesungen über Zahlentheorie* (1927)
  • Graduate students? Davenport, Heilbronn, Estermann, …
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S. / Trans. A.M.S. / Math Zeitschrift / ...*
  • Books? Landau’s *Vorlesungen über Zahlentheorie* (1927)
  • Graduate students? Davenport, Heilbronn, Estermann, …
  • Overseas visits?
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S. / Trans. A.M.S. / Math Zeitschrift / ...*
  • Books? Landau’s *Vorlesungen über Zahlentheorie* (1927)
  • Graduate students? Davenport, Heilbronn, Estermann, …
  • Overseas visits? / Forced migration?
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Journals? *Proc. L.M.S.* / *Trans. A.M.S.* / *Math Zeitschrift* / ...
  • Books? Landau’s *Vorlesungen über Zahlentheorie* (1927)
  • Graduate students? Davenport, Heilbronn, Estermann, …
  • Overseas visits? / Forced migration? Rademacher, James, …
Open Questions

- How was Hardy-Littlewood’s Circle Method disseminated?
  - Graduate students? Davenport, Heilbronn, Estermann, …
  - Overseas visits? / Forced migration? Rademacher, James, …

- What were the connections between these mathematicians?
HARDY-LITTLEWOOD
HARDY-LITTLEWOOD

Hilbert

Weyl

Kempner

Landau

Rademacher

Davenport

Estermann

Heilbronn

Germany

USA

UK
Open Questions

• How was Hardy-Littlewood’s Circle Method disseminated?
  • Books? Landau’s *Vorlesungen über Zahlentheorie* (1927)
  • Graduate students? Davenport, Heilbronn, Estermann, …
  • Overseas visits? / Forced migration? Rademacher, James, …

• What were the connections between these mathematicians?
Open Questions

- How was Hardy-Littlewood’s Circle Method disseminated?
  - Graduate students? Davenport, Heilbronn, Estermann, …
  - Overseas visits? / Forced migration? Rademacher, James, …

- What were the connections between these mathematicians?

- To what extent did a distinct British “school” of analytic number theory emerge, based on the use of the Circle Method?
HARDY-LITTLEWOOD

Hilbert
- Weyl
- Kempner

Landau
- Heilbronn

Dickson
- Rademacher

Hua Luogeng
- Estermann

James

Davenport

Vinogradov

Uspensky
A British “School”?

Klaus Roth  
(1925-2015)  

Alan Baker  
(1939-2018)
To Be Continued . . .