Cardano's Opus perfectum: an encyclopedia to extend algebra beyond abacus environments towards a transnational dimension

Veronica Gavagna
University of Salerno
In 1535 Cardano, aged 34, was asked to give lectures on geometry and arithmetics at the Piattine Schools, a College in Milano founded for the study of humanities and mathematics.

In 1539 he wrote *Practica arithmetice et mensurandi singularis*.

“Cardano had a sense that he needed to insert himself into the world of printed books ... he used the printing press to turn himself into a European figure” (Ian Maclean)

[Gavagna 1999], [Gavagna 2010]
Even though *Practica* was a book very close to an abacus treatise with respect the content (elementary operations, rule of three, collection of solved commercial problems, practical geometry and so on)

nevertheless it represents a breakpoint into the practical mathematics «local» tradition.
...

It's written in Latin, while the abacus tradition is a vernacular one.
- Writing mathematics in Latin was not properly a novelty, but...

Which was Cardano’s goal in choosing the Latin rather than the traditional vernacular?
- to overcome Pacioli’s fortune
- to increase and change readerships (he was convinced that one should write differently for different readerships)
Practica takes terminology from the abacus tradition; an awkward heritage as it presents the difficulty of translating some particular expression of the abacus vulgar lexicon in Latin terms.

Sometimes the solution to this difficult problem can be reduced to a simple adaptation, like in the case of the neologism schisatio (schisar i rotti) that is the representation of a fraction in reduced form, with numerator and denominator that are relatively prime.
According to Alfonso Ingegno, in *De sapientia* and *De subtilitate*, Cardano states the existence both of a divine (not accessible) knowledge and a natural knowledge.

The philosopher Cardano feels himself as the man who creates the conditions and the tools for the general improvement of a knowledge aimed to increase utility for all people. This is the reason why he set out to write a comprehensive account of astrology, of medicine, of natural philosophy (*De subtilitate*), and of geometry and arithmetics.
The first mathematical project
Commentaria in Euclidis Elementa

At the end of the Practica Arithmeticae we find this document, dated 1538, enumerating 34 unpublished works Cardano had asked the imperial press licence for.

[Gavagna 2003]
Searching for Euclid’s Commentary in *De libris propriis* (1544)

Cardano tells us that in 1535 he started writing a treatise about geometry. At first, this work consisted in a small book, but it soon became larger and larger. So he needed to divide it into three books, seven, nine and finally fifteen books. It contained about four hundred propositions, some of them by Cardano himself, some by other authors (among them, Archimedes’ *De spiralibus* and *De sphaera et Cylindro*, Apollonius' *Conica*, Ptolemy's *Almagestum*, Sacrobosco's *Sphaera*).

Cardano explains his ambitious project

*Euclides omnia collegit, quae in Geometria ab antiquis scripta essent, nos quae invenimus, suis autribus adscriptis, nostris inventis adiunximus, uno Archimede relictio, quem cum Archinto illi adiungere decrevimus* (Opera omnia, vol. I, p. 57).
Cardano gives more details about his geometrical treatise, named from this moment onwards, *Nova Geometria*. He explains that it deals with all Euclid's Elements books, except Books V, VII-IX, X. Cardano adds that his manuscript is not yet ready for press, because he has to complete the lacking sections, then reorganize and correct it.

In his autobiography, *De propria vita liber* (1576), Cardano mentions a list of unpublished writings in which features the *Nova Geometria*, not divided in fifteen books but only in two.
The National Library of France in Paris, conserves a manuscript attributed to Cardano, titled *Commentaria in Euclidis Elementa*.

It's a ponderous manuscript of 120 folios, full of corrections and additions written by the same hand, except some pages, clearly written by another hand (Ludovico Ferrari??)

The writing is incomplete, especially in the last part of the work, where a number of propositions are without any proof.
The manuscript is divided into nine books:

Books I to III are connected to the first four books of the *Elements*,
Books IV to VI are devoted to the application of theory of proportions on plane figures (i.e. Book VI of the *Elements*)
Books VII to IX concern geometry of solids.

I think that this manuscript may be an intermediate version, in nine books, of the *Nova Geometria*.

[Gavagna 2003]
Cardano’s Commentary and its connection with Ars Magna

Apparently, this geometrical Commentary is completely independent by Cardano’s algebraic writings....

But there are some interesting examples that remark the connections between Ars Magna and Commentaria in Euclidis Elementa.
About other (apparently) incomprehensible references in *Ars Magna*

*Ars Magna*, 1545, 1570, 1663, XI. *De cubo et rebus aequalis numero*


*Ars Magna*, 1570, 1663, XIII. *De cubo et numero aequalibus rebus*

h t, cumque numeri, ut ordinem et fit media proportione per 67. lib. de Prop. seu quinti huius inter f h & aggregatum f h & a d, erit quia f h & a d iunctæ sint prima quantitas s...
Hieronymi Cardani, Mediolanensis, C. Viro domino, medicinae, medicinae praecipuous, de Alze regus, Libellus, hoc est: " De Proportionibus, seu Operis Perfecti Liberi Quinti."

Prima definition.
Propositio ab Euclide se describitur, quod sit duarum quantitatum eodem genere, quod ad magnitudinem aequalis, comparatio certa.

Secunda definition.
Propositio per similitudinem dicatur, si quidem quantitati comparat alterius generis, cui sunt quotiesque eilia potestatis.

Tertia definition.
Propositio aequalis proposti sit, cum omne modo termini velut a b fingantur monas in comparatione ad c b e rectangulum a c e quale linea b c.

Hieronymi Cardani, Mediolanensis, C. Viro domino, medicinae, medicinae praecipuous, de Alze regus, Libellus, hoc est: "Opus perfectum in scriptu, esto in ordine Decimus."
What is *Opus perfectum*?

It is Cardano’s second mathematical project: an encyclopedia of arithmetics.

*De libris propriis, 1544*

The plan of *Opus arithmeticae perfectum* is outlined very clearly: strictly following the structure of *Practica arithmetice*, the fourteen books composing *Opus perfectum* would have needed to contain

- the fundamental calculations between integer, fractional, irrational and denominated numbers (books I-IV),
- proportions (book V)
- the properties of numbers (book VI),
- commercial arithmetics (books VII-IX),
- algebra (book X)
- plane and solid geometry (books XI-XII)
- arithmetic and geometric problems (books XIII-XIV)

[Tamborini 2003], [Gavagna 2010, 2012]
De libris propriis, 1557

Describing *Opus perfectum*, Cardano writes some *incipit* and we can recognize...

Book I: «*Si ab antiquitate aut necessitate*»
*Tractatus de integris*, OO.X.117-128

Book III: «*Cum in radicibus quantitatum*»

Book IV: «*Denominatio est numerus*»

Book VI: «*Numerorum alii dicuntur pri mi*»
*De numerorum proprietatibus liber unicus*, OO IV.1-12

Book X: «*Haec ars olim a Mahomete*»
*Ars magna*

[Maclean 2004]
De Nobilitate & utileitate huius artis & notis obscurioribus.

I ab antiquitate aut necessitate disciplina villa nobilis dicit potest, numerandi scientia omnium erit certe præstantissima, quippe quod eius origo in mundo fuit ante mundum, inter homines ante homines: illustrata verò his temporibus in Graecia, quibus nondum viribus, nec humani cultus extant, eratque Liberum Patrem, qui verusissimus est mercaturam primam dociisse, cum iam diu antea numerandi visus cognitus esset ut non solvam aut initiatum, sed ab initio earum Arithmeticae utilitas inchoauerit. Nescue immersum, nam cum cæteræ artes aliquidus & aliquando conveniant, Medicina ægis, Jurisprudentia literatoribus, Architectorum domo indigentibus, Militia pu-
In the *Tractatus de integris*, Cardano left the terminology of the abacus for a lexis more representative of the purity of Latin.

*sed nihil tam arduum quam vel nova novis rebus imponere nomina, quaedam vero obsoleta etiam in usum revocare ac in his omnibus Romanae linguae candorem retinuiisse*

For example, the term *irrationalis* is always preferred to *surdus*

the powers of the unknown, traditionally indicated with the terms *cosa, census, cubus, census census, relatum primum, relatum secondum* are respectively given the names of *positio, quadratus positionis, cubus positionis, quadratus quadrati, primum nomen, secundum nomen.*
HIERONYMI CARDANI,
DE NUMERORVM PROPRIETATIBVS
LIBER VNICVS.

CAPVT PRIMVM.
DE PROPRIETATIBVS VNIVS NUMERI
&S Secundi.Primo, De his quae ab Euclide describuntur
in Septimo & Octavo & Nono Libro
suorum Elementorum.

VMERORVM alij dicuntur
primi quia nulls alius nume-
rantur vt 7 &
43. alij com-
positi vt 3 &
alii nume-
rantur vt 2 &
j nam duco 1
in 5 lit 10. Hoc autem in tractatu de inte-
eris declaratum est. Compositorum quidam
quadra tus : confiat enim ex 3 & 3 & 36
vt confiat ex 6 in 6 est etiam quadratus,
vt vero confiat ex 3 in 12 est contentus lib
genere commun ; id est vocabitur superfici-
E t ita 5 est cubus confians ex 2 & 1
7 , cubus ex 5 & 3 & 3 ; confians
& 64 cubus , ex 4 & 4 & 4 , vt vero con-
flat ex 14.8. non est cubus sed solidi nomi-
ten tantum nuncupatur , vt vero confiat ex
& 2 dicitur superficialis , vt vero con-
flat ex multiplications 8 in 8 dicitur qua-
dratus from innumerus quandoque no-
ARS MAGNA, QUAM
VULGO COSSAM VOCANT, SIVE RE-
GULAS ALGEBRAICAS, PER D. HIERONYMUM
Cardanum in Quadraginta Capitula redacta, & est
Liber Decimus sui Arithmeticae.

De duabus æquationibus in singulis capitulis. Cap. I.

ÆC ari olim à Mahomete, Mosis Arabis filio initi-
um sumpsit. Etenim huius rei locuples testis Leonar-
tus Pisauriensis est. Reliquit autem capita quattuor,
cum suis demonstrationibus, quas nos locis suis as-
scribimus. Post multa uero temporum interuallum,
tria capita derivatis addit illis sunt, incerto aut
thore, quæ tamen cum principalibus, à Luca Pacciolo posita sunt.
Demum ex primis, alia tria derivationtius, à quodam ignoto iuro
inventa legi, hæc tamen minime in lucem prodierant, cum essent alii
longe utiliora, nam cubi & numeri & cubi quadrati æquatione do-
cebant. Verum temporibus nostris, Scipio Ferreus Bononiensis, ca-
pitulum cubi & rerum numero æqualium inuenit, rem sanè pulchram
& admirabilem, cum omnem humanam subtilitatem, omnis ingenii
mortalis claritatem ars hæc superet, donum profecto celeste, expe-
ri...
Concluding remarks

Cardano is mostly famous for its *Ars magna*: the mathematical papers published in the fourth volume of *Opera Omnia* appear as incoherent fragments.

The two encyclopaedic projects (*Nova Geometria, Opus arithmeticae perfectum*) could give an interpretation key to put in a coherent framework the great part of these papers.
Bibliography


